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The Emergence of Symbolic Algebraic Thinking in Primary School

Luis Radford

Abstract This chapter presents the results of a longitudinal investigation on the emergence of symbolic algebraic thinking in young students in the context of sequence generalization. The investigation rests on a characterization of algebraic thinking based on its *analytic* nature and a careful attention to the *semiotic systems* through which students express the mathematical variables involved. Attention to the semiotic systems and their interplay led us to identify non-symbolic and symbolic (alphanumeric) early algebraic generalizations and the students' evolving intelligibility of the variables and their relationships, and mathematical sequence structure. The results shed some light on the transition from non-symbolic to symbolic algebraic thinking in primary school.

1.1 Introduction

Over the years, the teaching and learning of algebra has consistently figured as one of the prominent research areas in mathematics education. Recently, research on early algebra has gained an increasing interest (see, e.g., Ainley 1999; Cai and Knuth 2011; Kaput 1998; Kaput et al. 2008b; Rivera 2010; Vergel 2015). Some of the main initial ideas behind the early algebra movement are to determine: (a) whether or not young students can really start learning algebra (Carraher and Schliemann 2007) and (b) if an early exposure to elementary algebraic concepts can alleviate the very well-known difficulties that adolescents encounter in algebra in secondary education (Blanton et al. 2017). Such ideas run against the traditional curricular conception that algebra can only be learned after the students have a sufficient knowledge of arithmetic, which excluded, until recently, algebra from primary school in many curricula around the world.

To merely envision the idea of exposing young students 5–12 years old to algebra instruction requires us, however, to revisit from new theoretical perspectives several key issues frequently discussed in the 1980s, such as the nature of algebraic thinking (Bednarz et al. 1996; Filloy and Rojano 1989; Kieran 1989a; Wagner and Kieran 1989). The efforts that have been made to come to terms with these and other concomitant issues have often led researchers to a sense of awareness that there are still many important things to investigate and learn. For instance, Carraher and Schliemann (2007, p. 676) remark that "the analysis of algebraic thinking is still in its infancy." And so is the analysis of the genetic relationship between algebra and arithmetic, and the role of signs in arithmetic and algebraic thinking.

Let us pause a moment and consider the role of signs. In his historical investigations, Damerow (1996) notes that ancient Egyptian and Babylonian arithmetic thinking arose as a result of *operating with signs* in order to systematically solve elementary problems involving counting and measuring. The earliest simplest numeric configurations were created by *iterating* a sign for the unit. These "numerals" had the purpose of facilitating the systematic calculation of additions and subtractions. As a result, from the outset, the concepts of addition and subtraction were consubstantial with the representations of the involved numbers. Embedded in practical activities oriented to the solution of administrative and other societal problems, the constructive-additive representation of numbers went hand in hand with the emergence of an elementary cognitive arithmetic additive structure. Later on we find the introduction of new signs to replace strings of sign-unit iterations and the ensuing rules of symbol-substitution, leading eventually to a positional numerical system. Within the possibilities of the historically developed cognitive additive-symbolic structures, new signs (e.g., signs for fractions) and operations (e.g., duplicating unit fractions, as in Egypt) became available. Contemporary elementary (school) arithmetic thinking depends no less on mathematical signs than the cultures of the past: it rests on a symbolic positional numerical system and sign-based algorithmic procedures for the basic arithmetic operations. A developmental approach to school arithmetic thinking would be impossible without attending to the role of signs. And so is the case of algebraic thinking in general, and early algebraic thinking in particular.

However, the exact role of signs in algebraic thinking remains a matter of contention among mathematics educators. In early algebra research it is not unusual (even if only implicitly) to see the use of alphanumeric symbolism as the trademark of algebraic thinking. Such a theoretical position is nonetheless untenable from a cultural-historical developmental viewpoint. The invention of alphanumeric symbolism is, indeed, a relatively historical recent event. It goes back to the work of 16th and 17th century mathematicians such as Rafael Bombelli, René Descartes, and François Viète. Equating the use of alphanumeric symbolism with algebraic thinking would amount to maintaining that algebra did not exist before the Western early modern period. Yet, 9th century Arabian mathematicians (like Al-Khwarizmi) and hundreds of Renaissance masters of abacus recognized and referred to their work as *algebraic*. So is the case of the 1544 “Libro e trattato della praticha d’alcibra” [Book and treatise of the practice of algebra] of the Sienese mathematician Gori (1984). You can go through the book page after page, line after line, word after word, and you will see no alphanumeric formulas or equations. You will see algebraic problem solving procedures expressed in words.

To better understand what can be termed as “algebraic” a more nuanced position is hence required. Mason et al. (1985), on the one hand, and Kaput et al. (2008a) on the other, offer a conception of algebra that is linked to the idea of generalization. For Mason et al.:

> Generality is the lifeblood of mathematics and algebra is the language in which generality is expressed. In order to learn the language of algebra, it is necessary to have something you want to say. You must perceive some pattern or regularity, and then try to express it succinctly so that you can communicate your perception to someone else, and use it to answer specific questions. (1985, p. 8)

Here, perceptual activity acquires a primordial role. They say: “Seeing, saying and recording form an important sequence in all maths lessons which applies particularly to all of the Roots of Algebra” (1985, p. 28). In this view, full symbolization—i.e., symbolization based on alphanumeric signs—is not required to start thinking algebraically: “Full symbolization should only come much later” (1985, p. 24).

Kaput et al. also link algebra to the expression of generalization: “We regard a symbolization activity as algebraic if it involves symbolization in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbol systems” (2008a, p. 49).

Although both perspectives on algebra revolve around the idea of generalization, they do not ascribe the same role to signs. While for Mason and collaborators the alphanumeric symbolism is not a condition for thinking algebraically, for Kaput and collaborators, in order for a symbolic activity to be called algebraic, full (i.e., alphanumeric) symbolization is required. Those activities in which generalization is expressed through other symbol systems are not considered genuinely algebraic: they are termed “quasi-algebraic” (Kaput et al. 2008a, p. 49). Along this line of thought, Blanton et al. argue that “algebraic reasoning ultimately involves reasoning with perhaps the most ubiquitous cultural artifact of algebra—the conventional symbol system based on variable notation” (2017, p. 182), which provides the rationale to attend to alphanumeric symbolism as early as Grade 1.

Perhaps we can better appreciate the differences between the aforementioned perspectives on algebra if we see them in terms of their conception about the role that signs play in cognition. Mason and collaborators’ perspective draws on an empiricist philosophy of language and symbols, one proponent of which was the 17th century British philosopher John Locke. For him, the relationship between cognition and signs is based on an epistemological schema that can be represented as follows:

Sensation → Ideas → Words

Within this schema, for Locke the purpose of language is to communicate ideas between individuals: "communication … is the chief end of language" (Locke 1825, p. 315). Within this context, "Words do not play a significant role in generating concepts since language enters the process post facto, after our ideas have been formed. Ideas come first: words follow" (Hardcastle 2009, p. 186). Or as Mason et al. say, "You must perceive some pattern or regularity, and then try to express it succinctly so that you can communicate your perception to someone else" (Mason et al. 1985, p. 8).

Kaput and collaborators also draw on an empiricist philosophy of language, but of a different kind—one that goes back to the 18th century French Enlightened tradition that had Étienne Bonnot de Condillac as one of its proponents. In Condillac's account signs are more than tools of communication: language and signs acquired a cognitive role in mastering human psychological functions.
Referring to memory and imagination, Condillac argued that

> by the assistance of signs he [the individual] can recall at will, he revives, or at least is often able to revive, the ideas that are attached to them. In due course he will gain greater command of his imagination as he invents more signs, because he will increase the means of exercising it. (Condillac 2001, p. 40)

We see that Condillac appears as a precursor of Vygotsky's concept of signs as mediators of psychological functions. It is precisely this concept of mediation that allows Kaput and collaborators to see a continuous connection between sign and ideas:

> Ideas, especially generalizations, grow out of our attempts to express them to ourselves and others, and our attempts to express them give rise to symbolizations that in turn help build and fill out the ideas, folding back into those ideas so that conceptualization and symbolization become inseparable. (Kaput et al. 2008a, p. 21)

One of the difficulties with the second perspective on algebra discussed above is the restrictive view that emanates from its requirement that thinking be expressed through the alphanumeric symbolism (or notations). I already mentioned that, from a cultural-historical developmental viewpoint, such a requirement may prove to be very limiting, in particular to approaches to early algebra. Such a requirement may lead to the failure to recognize non-symbolic forms of thinking as genuinely algebraic. Such a requirement may also lead to the attribution of an algebraic nature to forms of thinking that are in fact arithmetic. What is often overlooked is the fact that contemporary school arithmetic thinking resorts to alphanumeric symbolism too. The generalization $a + b = b + a$, which results from noticing that, for example, $2 + 3 = 3 + 2$, $1 + 6 = 6 + 1$, etc., may be considered as a genuine *arithmetic* generalization.

One of the difficulties with the first perspective on algebra was identified by Kaput et al.: "People have sometimes criticized inclusive views of algebraic reasoning on the grounds that it becomes difficult to distinguish thinking algebraically from thinking mathematically or (just plain) thinking" (2008b, p. xxi). Indeed, some researchers in the 1980s, like Kieran, expressed concerns about the difficulties of such an "inclusive" perspective: "For some authors (e.g., Open University 1985), the main idea of algebra is that it is a means of representing and manipulating generality and, thus, they see algebraic thinking everywhere—even in the recording of geometric transformations" (Kieran 1989a, p. 170). Certainly, by equating generalizing and algebraic thinking, it becomes difficult, if not impossible, to distinguish between an algebraic form of generalizations and other forms of mathematical generalizations (in particular arithmetic generalizations). As Kieran noted, "Generalization is neither equivalent to algebraic thinking, nor does it even require algebra" (1989a, p. 165). From research on animal cognition we know that chimpanzees, as well as birds, can start distinguishing between "edible" and "inedible" concrete items. They generalize their concrete experience and come to form what we humans would term the concept of "edible" (for details, see Radford 2011). Yet, we could hardly say that the chimps' generalization is an algebraic one.

To sum up, I have pointed out one difficulty arising from each one of the two perspectives on algebraic thinking that I have been discussing. An additional common difficulty is the fact that they reduce arithmetic thinking to mere computation. In other words, arithmetic thinking turns out to be reduced to procedural and mechanic calculation. I want to argue that this is a too restrictive view on arithmetic thinking. There are generalizations in arithmetic too. There may be very sophisticated arithmetic generalizations in the early grades that we are not even aware of, given the limiting view of arithmetic thinking that has been often adopted in early algebra research.

To move forward, we need to overcome the enduring conflation of algebraic thinking and notation use on the one hand, and the conflation of algebraic thinking and generalization on the other hand. Two points may be convenient to consider in this endeavor. First, notations are neither a necessary nor a sufficient condition for algebraic thinking (Radford 2014). Second, generalization is a common attribute of human thinking and cannot consequently capture the specificity of algebraic thinking. Our question is: What is it then that characterizes algebraic thinking?

The suggestion that I want to make draws from Kieran's (1989a) work on the one hand, and the work of Bednarz and Janvier (1996) and Filloy et al. (2007) on the other. I start from Kieran's 1989a paper and the idea that "For algebraic thinking to be different from generalization, I propose that a necessary component in the use of algebraic symbolism is to reason about and to express that generalization" (Kieran 1989a, p. 165). I want to make two points.

The first point: I want to take a very broad view on what counts as algebraic symbolism. In this view, I suggest that genuine algebraic symbolism includes the alphanumeric symbolism but also non-conventional semiotic systems—like natural language, which is mentioned in Kieran's paper, as well as gestures, rhythm, and other semiotic resources through which, as recent research shows, students signify generality (Radford et al. 2017).

The second point: There is something that remains unspecified in Kieran's proposal, namely what is meant by "to reason about and to express that generalization" (Kieran 1989a, p. 165). The reasoning that underpins the students' algebraic activity has to be specified. It cannot be *any* form of reasoning. It has to be *algebraic*. But what is *it*? It is at this point that I bring in the work of Bednarz and Janvier (1996) and Filloy et al. (2007). The Montreal team and the Mexican team have shown that one of the characteristics of algebraic thinking is its *analytic* nature (see, e.g., Bednarz et al. 1992; Filloy and Rojano 1989).

My suggestion is that algebraic thinking

- resorts to:
  - (a) indeterminate quantities and
  - (b) idiosyncratic or specific culturally and historically evolved modes of representing/symbolizing these indeterminate quantities and their operations,
- and deals with:
  - (c) indeterminate quantities in an *analytical* manner.

(a) Indeterminate quantities refer to the fact that the situation the students tackle in an algebraic manner involves more than given numbers or other mathematical entities. Indeterminate quantities can be unknowns, variables, parameters, generalized numbers, etc.

(b) As mentioned previously, although indeterminate quantities can be expressed through alphanumeric symbolism, they can also be expressed through other semiotic systems, without detriment to the algebraic nature of thinking. Naturally, alphanumeric symbolism constitutes a powerful semiotic system. With a very precise syntax and an extremely condensed system of meanings, alphanumeric symbolism offers a tremendous array of possibilities to carry out calculations in an efficient manner—calculations that may be difficult, if not impossible, to carry out with other semiotic systems (gestures, for instance, or even natural language). Yet, from an early algebra perspective, in the students' first contact with the historically evolved form of algebraic thinking conveyed in contemporary curricula, alphanumeric symbolism may not be required. The students can also resort to idiosyncratic or non-traditional modes of representing/symbolizing the indeterminate quantities and their operations.

(c) The indeterminate quantities and their operations are handled in an *analytic* manner. That is to say, although these quantities are not known, they are added, subtracted, multiplied, divided, etc. as if they were known—as Descartes says "without making a distinction between known and unknown [numbers]" (Descartes 1954, p. 8).

The adjective *analytic* comes from the noun analysis, which the ancient mathematician Pappus explained as the movement from what is given to what is sought (Rideout 2008). It is in this sense that algebra is considered by Viète as an analytic art where you make deductions; that is, you work from what is admitted "through the consequences [of that assumption]" (Viète 1983, p. 11). It is true that Viète introduced letters in a systematic way to solve problems algebraically. Certainly, he was aware of what he was accomplishing. Yet, he did not call his work “algebra with letters.” What was distinctively algebraic for him was something else: the analytic manner in which we think when we think algebraically. Hence, the title of his work is *The Analytic Art* (Viète 1983).

Let me consider the equation $2x + 2 = 10 + x$. In the perspective on algebraic thinking that I am outlining here, a solution by trial and error would not be considered as algebraic, even if the task includes indeterminate numbers and the students are working with notations. In a solution based on trial and error, the students are resorting to arithmetic concepts only. By contrast, if the students deduce from $2x + 2 = 10 + x$ that $2x = 8 + x$ (by subtracting 2 from both sides of the equation), etc., we can say that the students are thinking algebraically. They are working through the consequences of assuming that $2x + 2$ is equal to $10 + x$. Likewise, in pattern generalization, an algebraic generalization entails *deducing* a formula from some terms of a given sequence. That the formula be expressed or not in alphanumeric symbolism is irrelevant. Notice that the fact that the general term of the sequence be expressed in alphanumeric symbolism does not imply at all that the generalization is the result of thinking algebraically about the sequence. In Radford (2006), I discuss the way in which some groups of students tackle the generalization of a figural sequence made up of two rows (see Fig. 1.1).

The students resorted to a trial and error method: “times 2 plus 1”, “times 2 plus 2” or “times 2 plus 3” and checked their validity on a few cases. This form of thinking does not qualify as algebraic. Another group of students suggested: “$n \times 2$ (+3)”. When I asked how they arrived at their formula, their answer was: “We found it by accident.” Although the students’ way of thinking about the sequence involves indeterminate quantities and alphanumeric symbolism, the formula was not deduced, but guessed. This is an example of arithmetic generalization—a simple one. It is not an example of algebraic generalization.

The theoretical perspective on algebraic thinking that I present here might be of particular interest to early algebra research. Indeed, the criterion about analyticity—i.e., the specific *analytic* calculation with/on unknown quantities—offers an operational principle to distinguish arithmetic and algebraic thinking. The theoretical perspective recognizes the importance of the alphanumeric semiotic system, but does not confine algebraic thinking to it. It opens the door to the investigation of non-symbolic (i.e., non-alphanumeric) forms of early algebraic thinking. And it allows us to envision, under a new light, the educational problem of the transition from a non-symbolic form of algebraic thinking to a symbolic one. Some of my

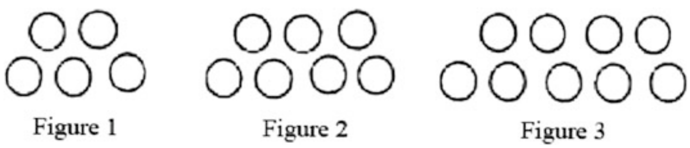

Fig. 1.1 The sequence of figures given to the students in a Grade 8 class (13–14 years old)

previous research has been devoted to the investigation of the emergence of early forms of non-symbolic algebraic thinking (Radford 2011, 2012). Focusing on pattern generalization, in this chapter I deal with the problem of the transition from non-symbolic to symbolic forms of algebraic thinking.

1.2 A Longitudinal Investigation of Early Algebraic Thinking

1.2.1 Research Methodology

The investigation that I report here was part of a six-year longitudinal research program in which Grade 2 students were followed as they moved from Grade 2 (7- to 8-year-old students) to Grade 6 (11- to 12-year-old students). In our research the primary interest is in understanding the development of students' algebraic thinking in situ. This starting premise is congruent with the fundamental principle of sociocultural research that stresses the link between cognition and context (Cole 1996). Drawing on the dialectic materialist theory of objectification (Radford 2008a), cognition can only be studied in *movement*; that is, through the activity in which it unfolds. In our case it is *classroom activity* (Radford 2015). As a result, our focus is the mathematics lessons.

We designed a flexible teaching-researching agenda committed to meeting two main goals. First, we sought to create the conditions that would allow the students to encounter the algebraic concepts stipulated by the curriculum. This was a practical concern framed by the political educational context of Ontario (Radford 2010a). Second, we wanted to deepen our understanding of the emergence and development of students' algebraic thinking, the difficulties that the students encounter as they engage in the practice of algebra, and the possible ways to overcome them. The longitudinal research was characterized by a continuous loop, which is represented in Fig. 1.2.

The arrows in Fig. 1.2 (and the whole Fig. 1.2) should not be understood in the empiricist sense of a clear-cut set of steps that assume that educational phenomena

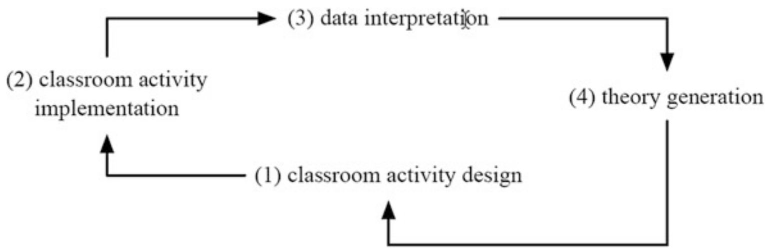

Fig. 1.2 Methodology of the longitudinal research (from Radford 2010a, p. 38)

obeys specific laws that are describable and whose variables can be controlled. In opposition to the Galilean paradigm of “teaching experiments,” we consider methodology as an inquisitional and reflective practice, a philosophical practice in fact, and adopt a social science paradigm that

> conceives of the educational phenomena as messy and context sensitive. [It is a paradigm whose] claims are not backed up by some immutable laws whose existence is asserted by a confrontation of the laws and empirical facts. Rather, general assertions are sustained by actual references that may guide further action through a reflective stance. (Radford and Sabena 2015, p. 158)

1.2.2 Data Collection and Participants

Our participants were 21 7- to 8-year-old students of a Grade 2 class in a public school in Sudbury, Ontario. In Grade 5 the class had 29 students and 31 in Grade 6. Data were collected through two one-week videotaped sessions per year, although we kept contact with the teacher during the year in order to exchange ideas and discuss the teacher’s and the students’ achievements and challenges about the teaching and learning of algebra. Each year, each one of the 10 videotaped lessons lasted 100 min. We had four cameras in the classroom to videotape a small group of students with each. In addition to the videotapes, we kept a copy of activity sheets, homework and individual written assessments (see below) of the videotaped groups as well as of the remaining groups of the class in order to broaden, complement, and enrich our videotaped data.

1.2.3 Task Design

Before each one-week videotaped session, the teacher and the research team (the author of this chapter and graduate and undergraduate students) participated in joint task design research meetings. The joint task design included a careful conception and production of

(a) pattern generalization problems for the students to solve in class,
(b) homework sheets, and
(c) individual written assessments.

During the joint task design sessions, videotaped classroom activities, transcripts, and copies of students’ sheets were discussed with the teacher (who changed from year to year) to highlight previous years’ students’ accomplishments and challenges.

1.2.4 Data Analysis

Our data analysis revolved around a multimodal approach that included fine-grained video-analysis (often short episodes subjected to frame to frame scrutiny) with special attention to gesture, language, perception, and symbol-use to account for non-conventional forms of signifying mathematical generality.

Problems of increased difficulty appeared as the students moved from grade to grade (for example, generalizations of figural sequences showing non-consecutive terms (e.g., Terms 1, 3 and 5); generalization of figural sequences where variables are organized in tables, and numeric (i.e., non-figural) sequences without geometric-spatial clues). Using the modern algebraic symbolism, almost all sequences corresponded to the formula $y = ax + b$ (with $a \in Z$, and $b \in N$). From Grade 3 on, a "core problem" remained invariable each year to better appreciate the students' progress. Because of space limitations and the fact that activities surrounding alphanumeric algebra appeared in Grade 4 for the first time, this chapter revolves mainly around this core problem and what happened in Grades 4, 5, and 6.

1.3 The Core Problem: "The Tireless Ant"

The core problem featured an ant that found a container with one crumb in it. The ant collected two crumbs each day, so that at the end of Day 1 the ant had 3 crumbs in the container; at the end of Day 2, it had 5 crumbs; at the end of Day 3, it had 7 crumbs, etc. A drawing (see Fig. 1.3a) was included in the activity sheet. Working in small groups of three or four, the students were invited to draw the container for Days 4 and 5, and then to find out the number of crumbs on Day 33. Then there was a question dealing with the writing of a message for another student. I shall return to the message question below.

The question of drawing the container for Days 4 and 5 was intended to investigate the students' evolving awareness of the mathematical structure of the sequence, and the semiotic means to which they resort to make the structure

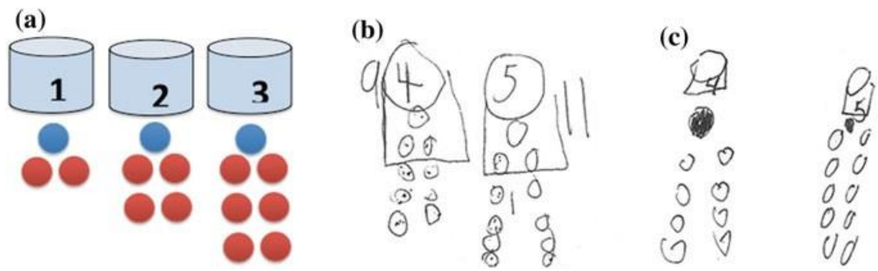

Fig. 1.3 The first terms of the sequence and examples from students' extension of the sequence

apparent. The question about Day 33 should reveal the students' actual generalizing process. The manner in which the students draw and talk about the content of the container on Days 4 and 5 provides us, indeed, with clues about their developing awareness of sequence structure.

1.4 Grade Four

Figure 1.3b, c show two typical answers of Grade 4 students. They come from one of the four small groups we videotaped. By looking at the drawings alone, there seems to be no difference. However, the video analysis shows that the underlying generalizing logic is not the same. Jay's drawing (Fig. 1.3b) is based on the recurrent relation mentioned in the statement of the problem: two crumbs are added each day. Jay says: "For Day 4 we draw the number again [i.e., Day 3]. After that we will add … [two crumbs]." He draws the crumbs by rows. Alex, by contrast, perceives the term globally. Visually, he recognizes figural *parts* of the term as *key parts* to make the drawings. Thus, after hearing Jay's utterance, he moves close to Jay's sheet and, pointing at the left column of Term 3, says: "There, there is 3 there at Day 3 (at the same time he counts successively the circles), plus (pointing at the initial crumb) the one on the top. So, we must always draw the number of days like this (pointing at the left column) plus one on top."

The "recurrent" and the "global" approaches (illustrated by Jay and Alex, respectively) are predominant in Grade 4. The first one is based on the recurrent relation between consecutive terms. The second approach goes beyond what is explicitly stated in the problem. It deals with the expression of a mathematical relationship between two variables: the number of the day and *visual key parts* of the term (the number of crumbs on the columns of the term). This approach requires a specific perceptual activity and a finer interpretation. Yet, we see Alex's difficulties to express verbally the key parts. They are referred to through pointing gestures. The awareness of the term structure seems to remain to a large extent visual: the perceived thing seems to remain inexpressible in the realm of language; it is hence expressed otherwise—by resorting to another semiotic system: the dynamic and frugal semiotic system of gestures. All in all, the grasping of the structure unfolds in a process of perceptual semiosis through language, gestures, the pictorial sign of terms, and visual activity.

But there is an additional point that needs to be discussed: the role of the temporal adverb "always" in the second part of Alex's utterance. "So, we must always draw the number of days like this (pointing at the left column) plus one on top." The temporal adverb "always" is what bestows the phenomenon under discussion with its full generality. What Alex has just perceived does not apply to Day 3 only. This is corroborated by the absence of specific numbers in the second part of Alex's utterance. Alex is not talking about Term 3 only. He is talking about *all* terms of the sequence. This is why, when the group moves to the question of drawing the container for Day 5, the question was quickly answered. Catherine said: “There are 5 on the side.” For the first time, the visual key part of the term is explicitly named. It is named the “side.” While naming the visual key part of the term, Catherine makes two straight sliding gestures, meaning the two columns of Term 5. It is as if the name alone is not enough to convey unambiguously its reference. Catherine resorts, hence, to gestures to complement the emerging meaning. At this point the teacher comes by to check on the students’ work. Jay has just finished drawing Day 5, still row by row. Talking to the teacher, Catherine addresses the question of Day 33, and quickly says: “So you put 33 and 33” while again making two sliding gestures. Taking into account the first crumb in the calculations, Catherine and Alex say “67.” Jay says “yes,” and switching to the global perception of the terms, adds: “It is the same number of all things.” Alex replies: “underneath each side there is the number of things, so 33 … plus 33 plus the one on top.”

1.5 Factual Generalizations

In the previous section we see the students noticing a structure in the first given terms (Days 1, 2, and 3), and generalizing it to all terms of the sequence. More precisely, the students started by grasping a *commonality* noticed on the first three given terms (Days 1, 2, and 3), which have been perceived as having “sides.” Then, the students generalized this commonality to all subsequent terms and were able to use the commonality to provide a direct expression of any term of the sequence. The generalized commonality is what Peirce (1958, 2.270) called an *abduction*—i.e., something only plausible. In the last part of the generalization process this abduction became the warrant to deduce expressions of remote elements of the sequence. Direct expression of the terms of the sequence requires the elaboration of a formula (that is, a *rule* or *method*) based on the variables involved. The analytic trait that, as I suggested above, is required for the generalization to be algebraic is to be found in the passage where Alex contends that “we must *always* draw the number of days like this (pointing at the left column) plus one on top.” The analytic trait is manifested in the *deduction* that Alex expresses in his utterance (as opposed to an induction). All things kept the same (i.e., the tireless ant always adding two crumbs each day), Alex can deduce that “underneath each side there is the number of things, so 33 … plus 33 plus the one on top.” Although the students have not used alphanumeric symbolism, the students’ generalization is genuinely algebraic in nature.

I have taken some time to analyze the students’ generalization, as it shows an example of algebraic generalization that is not based on the alphanumeric symbolism. In previous work I have called this type of generalization *factual generalization* (Radford 2011). The adjective *factual* means that the variables of the formula appear in a *tacit* form. The formula is expressed through particular instances of the variable (the variable is instantiated in specific numbers or “facts”) in the form of a *concrete rule* (“33 plus 33, plus the one on top”). This concrete rule empowers the students to deal with any specific term of the sequence (e.g., Terms 100, 500). To make sure, I came to see the group and asked about Day 60. Catherine answered: “we would do 60 on one side, 60 on the other …” Alex interrupted and added: “and the one on top.”

1.6 Writing a Message: Contextual Generalizations

Our path towards symbolism was based on the following question: The students were asked to write a message for another student to tell her how to quickly calculate the number of crumbs in the container for a certain day. The number of the day was drawn from a box containing cards, each one with a big number on it. First, the teacher drew a card and showed it to the students. The card had the number 100 on it. Here is an excerpt of the discussion in Alex’s group.

1. Alex: We put the number on both sides … and, and one on top and add all that (he writes)
2. Teacher: I am going to read (reading) “We put the number on both sides.” Which numbers?
3. Alex: The number of the day … (talking to his group-mates) we write the number on both sides of the day, write the number of the day.
4. Jay: And you have to add both days.
5. Catherine: (Interrupting) Together.
6. Jay: Ah yeah! OK it’s like … one must add … the two days together and add another … another day!
7. Alex: I don’t get it. I do not understand what you are saying … no offence man, but …

In Turn 2, the teacher asks the students to specify which numbers they are talking about. In Turn 6, Jay mixes the number of crumbs and the number of days. This dialogue highlights some of the difficulties that the students found in articulating in a clear manner the variables and their relationship at the level of language. These difficulties appeared also in the students’ written messages. The messages were essentially of the same form: a drawing with some calculations and a short text. Figure 1.4a, b show a paradigmatic drawing and a text. In the drawing, the student identifies the container as “Jour 100” (Day 100). He explains that the black circle is the initial crumb (“miette”). Towards the right of Fig. 1.4a, he writes: “One adds 100 on each side.” The text brings forward the spatial context in an explicit manner (see Fig. 1.4b). It reads: “One remarks 100 on each side. One adds it to arrive to the answer and one adds the crumb that he found. At the end one makes a calculation. Here is how to solve this problem.”

We see that in both the student’s written text and the oral discussion (see previous excerpt), the relationship between the variables remains unclear. It is as if the formula has not yet completely entered the realm of verbal thinking.

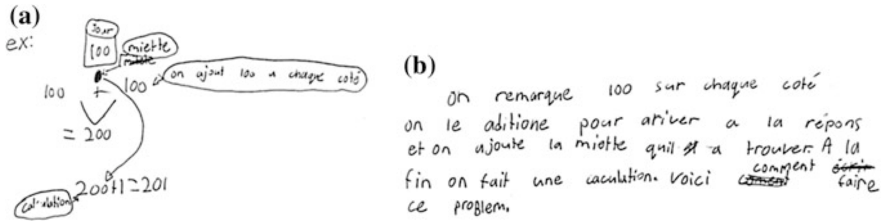

Fig. 1.4 A paradigmatic example of the student's written text

Yet, despite the challenge of putting the formula in words, the dialogue above shows that the students have moved to a new layer of generality. Although the object of discourse in the previous dialogue is Term 100 of the sequence, we see the students engaged in a discussion where numbers start receding to the background (see the students' dialogue above). The students' attention moves to the variables and their relationship, which, bit by bit, become the central object of discourse.

1.7 The Emergence of Symbolic Algebraic Thinking in Grade Four

After a general discussion, the teacher again drew a card from the box but hid the number on it from the students. She said: "I draw a card, I do not tell you the number. I put the card in an envelope and I will send it to a student. What do you write in the message to this student now?" Marika replied: "As Catherine and Alex said, it is twice in the line [i.e., the "side"], it is the same thing day and crumbs, and you have to put twice, and then you have to add the crumb on top that it [the ant] has already found."

Marika's utterance offers an example of a *contextual generalization*. That is, a generalization whose formula is based on spatial and other deictic terms (here, "sides" and "top"). The deictics endow the variables with a meaning deeply related to the spatial or other contextual clues of the terms of the sequence (Radford 2011).

After a general discussion about Marika's formula, the teacher moved towards the didactic agenda: the search for a symbolic formula. The teacher said: "Now I do not want you to put a phrase. I want you to write down a calculation." Dylaina suggested to use a letter, but formulates the message as if the number was known: "You put r for the number of days and you put on each side and it is equal to 200, then you add 1 and it is equal to 201." The teacher reminded the students that the number in the envelope is not known. A student went to the blackboard and suggested to use the sign "#" for the number of the day; other students suggested the signs "?" and "_." The students' formula on the blackboard was: 2 × __ = + 1 =_ (see Fig. 1.5a). The teacher asked if they could use letters instead. The students suggested *a*, *b*, and *c*, so the formula was transformed into

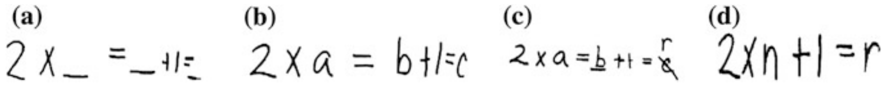

Fig. 1.5 The first symbolic expressions in Grade 4

2xa = b + 1 = c (see Fig. 1.5b). The 100-minute math lesson ended up with the teacher asking the students to reflect on the meaning of each letter.

The next day the class came back to the formula 2×a = b + 1 = c. Since the last number is the answer (“réponse” in French), the students suggested replacing “c” with “r” (see Fig. 1.5c). The teacher started a new thread in the conversation.

1. Teacher: I will write something on the blackboard and I want you to tell me if I can do this (she writes on the blackboard; see Fig. 1.5d).
2. Students: Yes!
3. Teacher: I need someone to explain … Lola, would you like to explain?
4. Lola: Because 2 times the number plus 1 equals the answer.
5. Teacher: Ok. And *n*, what does it represent?
6. Lola: It represents 100, 101, etc.
7. Teacher: Ok. And plus 1, what does it represent?
8. Lola: It represents the first crumb.

The teacher then asked if other formulas were possible. Alex suggested: “*n* plus *n* equals plus 1 equals *r*.”

Generally speaking, the class made substantial progress towards the production of alphanumeric formulas. However, although the produced formulas start moving away from the recourse to the spatial deictics that are the hallmark of contextual generalizations, the formulas exhibit something that manifests itself as one of the greatest obstacles in becoming fluent with the alphanumeric symbolism and the meaning of the symbolic formula, namely the strong tendency to *calculate sub-totals*. The alphanumeric formula expresses the algebraic calculations in a *global* manner. It focuses on the structure. The students’ tendency to calculate sub-totals reminds us of Davis’s (1975) “process-product dilemma” (Kieran 1989b, p. 41). The “process-product” dilemma refers to the difficulty in considering an expression such as “x + 3” as an answer. In our interpretation, what this dilemma means is that the emphasis in the alphanumeric formula is not on the numbers themselves but *on the operations*. We move here to an altogether new realm of generality—*symbolic generality*. In this level of generality, the novelty is not only the introduction of alphanumeric symbolism, but a whole reconceptualization of numerical operations.

1.8 Grade Five

In Grade 5 the students again tackled the Ant Problem. This time the mathematical structure was easily perceived:

1. Catherine: So, we can do the first crumb first … the crumb he found first …
2. Alex: (Interrupting) And then there are 4, 4 on each side (he makes two gestures in the air meaning the two sides), [and] 1.
3. Catherine: Ah yeah, because the number of the … day equals the ones on the two sides … So 4 crumbs on each side …
4. Alex: And then, for 5, it's 5 on each side.

Alex mentions right away the spatial deictic "side," which is also the reference of his gestures. Although not yet perfect, the linguistic relationship between the variables is much better articulated than in Grade 4 and the mathematical structure of the terms is much better ascertained. And as the variables' relationship enters the realm of language, the gestural activity recedes into the background. This is why, when the students tackled in Grade 5 the question of Day 33, gestures were no longer required. The answer came without difficulty. Alex said: "So 33 plus 33 equal 66, plus 1."

This passage provides us with a neat example of *semiotic contraction*; that is, the mechanism that consists of making a choice between what counts as relevant and irrelevant. In semiotic contraction there is a reorganization of the semiotic recourses that help the students to direct their attention to those aspects that appear to be most significant. In general, semiotic contraction is an indicator of a deeper level of consciousness and intelligibility (Radford 2008b).

The students' deeper level of consciousness and intelligibility of the mathematical structure of the sequence was also manifested in the flexibility and creativity that the students showed in dealing with the ant context. Alex challenged his teammates Catherine and Andrew (who joined Alex and Catherine's group in Grade 5, while Jay went to work with another group), with the question of finding the crumbs in day 103: "OK. And if it is the 103rd day, how many pieces [crumbs] is he going to have?" Andrew replied immediately: "207." Andrew went even further and said: "Um, anything ["n'importe quoi"] plus anything equals um" He explains: "I do it the other way: I give you the answer, but I do not give you the numbers."

During a general discussion, the teacher visually and discursively emphasized the relationship between variable "Day" and "number of crumbs" in the container. The teacher said: "If I want to draw Day 6, I have to put one crumb (she draws 1 crumb) and how many circles should I put here on my left column? (She makes a vertical sliding gesture where the crumbs/circles will be drawn).

After that, the group came back to Andrew's challenge.

1. Catherine: OK, so, it is day 201, no … there are 201 crumbs. Which day is it?"
2. Alex: 100! Woo!
3. Catherine: Now, challenge me!

During a series of consecutive challenges that the students enjoyed very much, they came to realize that the challenging number had to be an odd number. Also important from a developmental viewpoint is the fact that in the course of these challenges, for the first time, the students linguistically identified the variables in an explicit and proper manner. The consequence was that the contextual generalizations that they were producing were much more refined than those produced in Grade 4.

When it came to write the message as a sequence of operations, the students resorted to similar symbolic formulas as those they proposed in Grade 4; that is, formulas that include sub-totals. During the general discussion, another student, Janelle, wrote on the blackboard the equation that the class came up with last year: ___ × 2 =____+ 1 =______. Figure 1.6a, b show two more examples.

These figures suggest that students have become more and more conscious that different signs are required to represent different numbers. Thus, in Fig. 1.6a, Gavin explains: “We did a multiplication. We did the mysterious number times 2, equals a mysterious number, and there (pointing at the “?” sign on the second row; see Fig. 1.6a) plus 1 equals a mysterious number. Now, this (encircling the second and the third “?” signs) are the same thing, the same number.” Figure 1.6b is even more explicit about the fact that each sign stands for a different number. In Fig. 1.6c, the teacher invites the class to use letters. Alex suggests using “a,” “l,” and “e” (the first three letters of his name), while Christiane suggests “n” for number, “r” for “réponse” (i.e., answer), and “vr” for “vraie réponse” (i.e., the real answer). To close the 100-minute lesson, the teacher asked the students if they had learned something new. Théo answered: “We can put any letter as long as the numbers are different … ‘Cause that [same letters] means that the number is the same. You can use the same letter if it is the same number.”

The students had homework to return the next day. The homework featured the Tireless Ant context with 2 crumbs found in the container and drawings of the container for Days 1, 4, and 5 (Day 1 = 5 crumbs; Day 4 = 14, and Day 5 = 17). Of the 26 students, 1 student did not return the homework, 21 students resorted to formulas based on sub-total calculations (e.g., $a \times 3 = b + 2 = c$), 3 formulas conforming to the alphanumeric syntax (e.g., $J \times 3 + 2 = R$), and 1 unclassified answer. The teacher started the lesson with a discussion of the students’ homework answers. Three students volunteered to write their formulas on the blackboard. One formula was: *number of the day* × 3 =___ + 2 = ____. The second formula was: $a \times 3 = b + 2 = c$; the third one was J 3×+ 2 = R. The teacher took the opportunity to make a distinction between the formulas. The first two, she insisted, “separated” the calculations into two. The third formula did not. She insisted that it was not necessary to separate the calculations. Then the lesson continued with

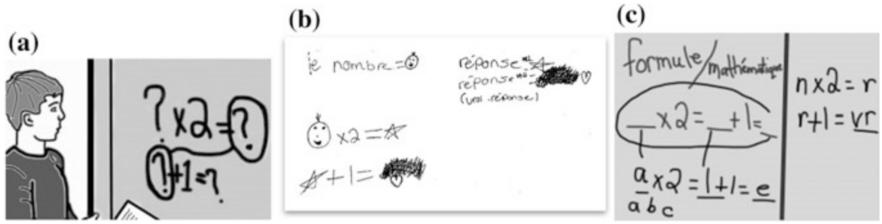

Fig. 1.6 Symbolic formulas in Grade 5 are still based on the calculation of sub-totals

another activity where the students were put in a position of formula interpretation. The activity included the formulas “$5 \times n + 2 = r$,” “$3 \times n + 7 = r$,” and a third question where the students had to produce their own formula. In each case they had to explain the meaning of the letters and coefficients within the context of days and crumbs. In general, the students were able to correctly identify the various terms of the formula. For instance, Miguel produced the formula $10 \times n + 5 = r$, and noted that 10 is the “added crumbs,” n is “the number of the day,” 5 is “first crumb,” and *r* the “answer.” An individual test took place two weeks later. By then, half of the 26 students were producing formulas conforming to the alphanumeric syntax, 7 students were producing formulas showing partial calculations and 5 students were producing other answers.

1.9 Grade Six

In Grade 6 the Tireless Ant activity did not include the drawings of the container and the crumbs/circles. Yet, the students were able to answer the questions quickly. Laura, for instance, said: “First, you have to take the number of the day and to multiply it by 2, because each day it [the ant] adds two crumbs. And you have to add 1, which is the crumb that the ant found in the container.” The symbolic formula was easily reached too. It read: $(n \times 2) + 1 = m$. The appearance of brackets in the students’ formulas was the result of a class discussion conducted by the teacher about the priority of operations.

During the activity concerning the interpretation of formulas, the teacher came to see Laura’s group and challenged the students’ interpretation. The formula under discussion was $5 \times n + 2 = r$. The students argued that the ant found two crumbs in the container and added 5 each day:

1. Teacher: Why not the ant started with 5 and added 2 each day?
2. Laura: Because times (pointing to the multiplication sign) means that [the ant] adds 5 each day, like 5 *times* the day… (she emphasizes the word “times”).

The following day, the students explored the sequence shown in Fig. 1.7a. The students were at ease producing a symbolic formula for the general term of the sequence. Alex, for example, suggested $n \times 2 + 2 = r$.

The activity included the following formulas: “$N + N + 1 + 1 =$ __” and “$2 \times N + 1 + 1 =$__” (which were actually produced by students of another Grade 6 class). The teacher asked her students whether or not they thought that these formulas were correct and to explain. Referring to the first formula, Christiane answered: “Yes. N = number of the figure; ____ = number of rectangles in total; 1 = the rectangles added on the top, at the ends.” Referring to the second formula she noted: “Yes. N = number of the figure; = total of rectangles; 1 = the rectangles added on top, at the ends; 2 = two rows of rectangles.” Another question

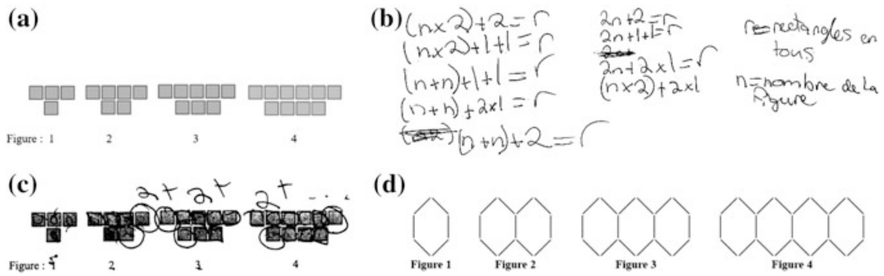

Fig. 1.7 Figural sequences and symbolic generalizations in Grade 6. a (left, top row) shows a figural sequence investigated in Grade 6. b (right, top row) shows Christiane’s formulas. c (left, bottom row) shows traces of the students’ perceptual and symbolic activity. d shows a sequence in the individual test

asked the students to produce as many formulas for the sequence as they could. Figure 1.7b summarizes Christiane’s results.

As Fig. 1.7c suggests, the students have reached a very good coordination of perceptual and symbolic activity. Indeed, the students were able to interpret perceptually the given formulas “N + N + 1 + 1 = __” and “2 N + 1 + 1 =_”. Figure 1.7c shows some traces of the students’ perceptual activity: the two added rectangles were imagined at the right end of the rows; but also at the beginning of the rows, and also one at the beginning of the bottom row and the end of the top row. It is not the symbolic function that has evolved, but mathematical imagination and perceptual activity as well. The eye appears now as a theoretician (Radford 2010b).

Let us pause for a moment and discuss with more detail Fig. 1.7b. To produce the formulas in Fig. 1.7b, the students did not need to see the terms of the sequence, nor did they have to translate the meaning of the formulas and their components in natural language according to the context (i.e., the students did not need to refer to *n* as, e.g., “the number of rectangles on the bottom row”). Letters, constants, coefficients, and operations were uttered in a transliterated form only (e.g., “two *n* plus two equal *r*”) or were not uttered at all. What this means is that, remarkably, for the first time in the students’ mathematical experience, as we witnessed in the course of our longitudinal investigation, natural language was no longer leading thinking. At this precise point in the development of the students’ algebraic thinking, abstract signs (what Peirce (1958) called *symbols*, i.e., abstract signs vis-à-vis the context) starting leading and words started following. In other terms, symbolic thinking has superseded verbal thinking!

To end this chapter, it might be important to say something about the results of an individual test in Grade 6. The test took place two days later. It included the sequence shown in Fig. 1.7d. The formulas with sub-totals disappeared completely. In the course of the years, with the help of the school Principal, we managed to keep most of the students in the same class. But in Grade 6 there were a few newcomers and two students moved to other schools. We lost Catherine and another student. Of the 31 students in our Grade 6 class, 21 students produced the expected alphanumeric formula—usually $n \times 5 + 1 = r$ or $(n - 1) \times 5 + 6 = r$. Ten students produced an incorrect formula for the problem. Of the 21 students who produced the expected alphanumeric formula, 19 were part of the cohort followed in this study and two students joined the cohort in Grade 5.

1.10 Concluding Remarks

In this chapter I presented the results of a longitudinal investigation on the emergence of algebraic symbolism in the context of sequence generalization. The investigation rests on a characterization of algebraic thinking based on its *analytic* nature. In sequence generalization, this idea means that the sought-after formula is not guessed but *deduced* from certain given data. In the course of the chapter I insisted that the formula does not necessarily have to be expressed through the alphanumeric symbolism. The formula can also be expressed through other kinds of semiotic systems.

The importance of distinguishing the semiotic systems through which the students produce their formulas is related to a dialectic materialist epistemological premise about cognition and signs, implicit in the Introduction, and that I can now state in full as follows. *The manner, depth, and intensity in which an object appears as an object of consciousness are consubstantial with the semiotic material that makes possible for such an object to become an object of consciousness and thought*. There are always limits to what can be thought and said within a semiotic system. For each semiotic system has its own *expressiveness*. In terms of Locke's and Condillac's philosophy of language and signs mentioned in the Introduction, what the dialectic materialist epistemological premise means is that language and semiotic systems in general are not merely the expressions of thought or mediators of it. As Vološinov notes, "It is not experience that organizes expression, but the other way around—*expression organizes experience*. Expression is what first gives experience its form and specificity of direction" (1973, p. 85; emphasis in the original). The alphanumeric symbolism and the Cartesian Graph symbolism, for instance, do not have the same expressiveness. There are inherent limits as to what can be said and thought within each one of them. Each one provides the students' experience of algebra with different form and direction.

Of course, this question about expressiveness is—reformulated at a more general level—the formidable problem that Vygotsky (1986) dealt with in the last chapter of *Thought and Language*. What our dialectic materialist epistemological premise means in the context of this chapter is that the conceptual deepness of the manner in which the variables and their relationships are noticed and the algebraic structure is revealed to the students is not the same in the various types of generalizations that we have discussed. In factual generalizations the formula is not expressed explicitly. It appears "in action," through concrete numbers and their operations. The variables and the relationship between the variables remain implicit. In contextual generalizations, by contrast, the formula is expressed at a more general level; the variables and their relationship become explicit and are referred to through contextual elements—spatial linguistic deictics (for example, “top” and “bottom”). While factual generalizations seem to go without difficulties in Grade 4, contextual generalizations were difficult to express. These difficulties reveal the students’ agony in coming to terms with a deeper level of algebraic structure consciousness. In Grade 5 things changed. The linguistic formulation of variables and their relationship became possible, the result being a deeper level of intelligibility. The algebraic formula entered the realm of *verbal thinking*. But it did not yet enter the realm of *symbolic thinking*. For this to happen, the teacher and the students had to continue working to achieve something that has far-reaching epistemological consequences. That is, as paradoxical as it may seem, the teacher and the students had to move to a conceptual realm where natural language ceases to be the main substance and organizer of thinking. Indeed, while natural language with its arsenal of conceptual possibilities offers the semiotic material to produce contextual generalizations, natural language has to recede into the background to yield space to a new cognitive form—symbolic thinking. The “deicticity” of contextual generalizations does not disappear: it becomes sublated into the new abstract signs of symbolic generalizations (see Fig. 1.6c).

Symbolic generalizations are, indeed, based on *symbols*—i.e., abstract signs vis-à-vis the context (see, e.g., Fig. 1.6b). The deictic nature of contextual generalizations may be formulated as an indexical and iconic form of signifying. The index and the icon signs (in Peirce’s sense) now have to recede for the symbol to appear. And Grade 6 was the moment in which this happened: it was the remarkable moment in which algebraic symbolic thinking emerged. The students had to overcome their tendency to think of the formula in terms of sub-total calculations (a symptom of the entrenched leading role that numbers *qua* concrete numbers had in the students’ thinking). Finally, this tendency disappeared and the focus shifted to variables, operations, and numbers reconceptualized at a higher level (as rates, for instance, as in the case of the multiplicative coefficient of linear formulas).

But we should not miss the point about the importance of the standard algebraic symbolism. The importance of the standard algebraic symbolism does not reside in its tremendous efficiency to carry out calculations only. It also resides in the possibilities it offers to reach new aesthetic modes of imagination and perception.

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